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Spontaneous breakdown of Lorentz invariance in IIB matrix model

Jun Nishimura* **and Graziano Vernizzi**

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
E-mail: nisimura@nbi.dk, vernizzi@nbi.dk

Abstract: We study the IIB matrix model, which is conjectured to be a nonperturbative definition of superstring theory, by introducing an integer deformation parameter ν which couples to the imaginary part of the effective action induced by fermions. The deformed IIB matrix model continues to be well-defined for arbitrary ν, and it preserves gauge invariance, Lorentz invariance, and the cluster property. We study the model at $\nu = \infty$ using a saddle-point analysis, and show that ten-dimensional Lorentz invariance is spontaneously broken at least down to an eight-dimensional one. We argue that it is likely that the remaining eight-dimensional Lorentz invariance is further broken, which can be checked by integrating over the saddle-point configurations using standard Monte Carlo simulation.

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*Permanent address: Department of Physics, Nagoya University, Nagoya 464-8602, Japan

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Recent developments in string dualities have suggested that the known five types of superstring theories in ten dimensions and M-theory in eleven dimensions are just different microscopic descriptions of the same underlying physics. This means that if any of the descriptions can be defined nonperturbatively, it is as good as any other to study the dynamics of the universality class of string/M-theories. A celebrated first step towards this end has been made by ref. [1], where the so-called Matrix theory has been proposed as a nonperturbative definition of M-theory in the infinite momentum frame. Soon after, the IIB matrix model [2] has been proposed as a nonperturbative definition of type IIB superstring theory. Although there are many evidences that support these conjectures, there is no direct proof that these models reproduce the string/M theories perturbatively.[1] If these conjectures are true, there is a hope to understand all the fundamental questions about the Standard Model, including the space-time dimensionality, gauge group, matter contents, the hierarchy problem, the cosmological constant problem, and so on, in terms of the dynamics of these models.

So far, the understanding of the vacuum of these models is quite limited. In this Letter, we attempt to extract information about the dimensionality of the space-time which is dynamically generated in the IIB matrix model. The IIB matrix model is a supersymmetric matrix model obtained formally by taking a zero-volume limit [4] of ten-dimensional SU(N) super Yang-Mills theory. Unlike in field theories, the space-time is treated in this model as a dynamical object, which is represented by ten bosonic $N \times N$ hermitean matrices, where N should be sent to infinity eventually. Therefore, the model has a potential to explain the dynamical origin of the dimensionality of our space-time, which, in the Standard Model, has to be given as an input parameter. If we are ever going to explain our four-dimensional space-time in this way, the vacuum of the IIB matrix model should be dominated by configurations which have only four-dimensional extent.[2] This means, in particular, that the ten-dimensional Lorentz invariance of the model should be spontaneously broken to a four-dimensional one.[3]

Such an issue can *in principle* be addressed by performing Monte Carlo simulation, since the model is completely well defined for arbitrary N without any cutoff [7, 8, 9]. Indeed the large-N dynamics of the four-dimensional version of the IIB matrix model, which can be obtained by a zero-volume limit of four-dimensional super Yang-Mills theory, has been understood to a considerable extent through Monte Carlo simulation [10]. Comparison of the results with those for the corresponding bosonic theory [11], revealed that supersymmetry indeed plays an important role in the dynamics of large-N reduced models as a nonperturbative definition of string theories. Unfortunately, such a direct approach does not work in the IIB matrix

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[1] See, however, ref. [3] as an attempt in this direction.

[2] This is similar in spirit to a description of our four-dimensional space-time as a "brane" in a higher-dimensional *non-compact* space-time, which is proposed [5] as an alternative to a more conventional compactification approach.

[3] The possibility of a non-trivial vacuum structure in string theory has been considered also in [6].

model as it stands, due to the fact that the effective action induced by fermions is generically complex, whereas in the above mentioned four-dimensional version, it is real. In general, when the action of a theory has a non-zero imaginary part, the number of configurations needed to extract any information increases as exponential of the system size. This is the notorious sign problem, which occurs also in many other interesting systems related to particle physics, such as theories with a chiral fermion (as in the present case), theories with a θ vacuum, Chern-Simons theories and theories with a finite baryon number density. We emphasize, however, that the problem is a purely technical one, and indeed its complete solution in some particular class of systems has been obtained recently [12].

In order to search for a way out of this difficulty, let us consider a deformation of the IIB matrix model by introducing an integer parameter ν, which couples to the imaginary part of the effective action induced by fermions. The deformed model, which reduces to the IIB matrix model at $\nu = 1$, is well-defined for arbitrary ν. It preserves both gauge invariance and Lorentz invariance, and moreover, it preserves the cluster property, which is an important consequence of the supersymmetry of the original model. At $\nu = 0$, the sign problem disappears and standard Monte Carlo techniques become applicable. Monte Carlo studies up to $N = 512$ show, however, that the ten-dimensional Lorentz invariance is *not* spontaneously broken [13], which suggests that if Lorentz invariance is spontaneously broken in the original model, the imaginary part of the effective action must play a crucial role. This motivated us to consider the opposite extreme, namely $\nu = \infty$.

At $\nu = \infty$, the integration over the bosonic matrices is dominated by the saddle-points of the imaginary part of the effective action. We find that the saddle-points are given by configurations which have only eight-dimensional extent. This implies that the ten-dimensional Lorentz invariance *is* spontaneously broken at least to an eight-dimensional one. Since we still have to integrate over the saddle-point configurations, the question arises whether the remaining eight-dimensional Lorentz invariance further breaks down, say, to a four-dimensional one. To this end, we study the hessian at configurations which have only d-dimensional extent ($d \le 8$) and find that it is zero of order $(8-d)\{2(N^2-1)-16\}$. In other words, the imaginary part of the effective action becomes *more* stationary for smaller d. This gives a huge enhancement to configurations with smaller d, which is found to cancel exactly the entropical barrier against having such configurations. Considering the effect of the real part of the effective action, which is studied in ref. [8] using perturbation theory, it is plausible that the vacuum of the model at $\nu = \infty$ is actually given by $d < 8$. We also find a lower bound on d as $d > 2$. Further information about the vacuum can be obtained by performing explicitly the integration over the saddle-point configurations by Monte Carlo simulation. Naively, one might consider that the sign problem, which already exists in the original model ($\nu = 1$), becomes maximally severe at $\nu = \infty$. This is not the case, as we will see.

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The IIB matrix model is formally a zero-volume limit of ten-dimensional pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory. The action, therefore, is given by

$$\begin{aligned} S &= S_b + S_f \\ S_b &= -\frac{1}{4}\,\mathrm{tr}\left([A_\mu, A_\nu]^2\right) \\ S_f &= -\frac{1}{2}\,\mathrm{tr}\left(\psi_\alpha(\mathcal{C}\Gamma_\mu)_{\alpha\beta}[A_\mu, \psi_\beta]\right). \end{aligned} \tag{1}$$

A_μ ($\mu = 1, \ldots, 10$) and ψ_α ($\alpha = 1, \ldots, 16$) are $N \times N$ traceless hermitean matrices, which can be expanded in terms of the generators t^a of $\mathrm{SU}(N)$ as

$$(A_\mu)_{ij} = \sum_{a=1}^{N^2-1} A_\mu^a\,(t^a)_{ij}\,; \qquad (\psi_\alpha)_{ij} = \sum_{a=1}^{N^2-1} \psi_\alpha^a\,(t^a)_{ij}\,, \tag{2}$$

where A_μ^a is a real variable and ψ_α^a is a real Grassmann variable. We have made a Wick rotation in the action (1), so that the metric has euclidean signature. The 16×16 matrices Γ_μ are ten-dimensional gamma matrices after Weyl projection, and the unitary matrix $\mathcal{C}$ is a charge conjugation matrix satisfying

$$\mathcal{C}\,\Gamma_\mu\,\mathcal{C}^\dagger = (\Gamma_\mu)^\top\,; \qquad \mathcal{C}^\top = \mathcal{C}\,. \tag{3}$$

The model has a manifest ten-dimensional Lorentz invariance, by which we actually mean an SO(10) invariance. A_μ transforms as a vector and ψ_α transforms as a Majorana-Weyl spinor.

Pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory can be also defined in 3D, 4D and 6D, as well as in 10D. Hence, by taking a zero-volume limit of these theories, we can define supersymmetric large-N reduced models, which are $D = 3, 4, 6$ versions of the IIB matrix model. A nontrivial question then concerns whether the integration over the bosonic matrices is convergent, since the integration domain for hermitean matrices is non-compact. A potential danger of divergence, even for finite N, exists when the eigenvalues of A_μ become large. This issue has been addressed in ref. [8] using a one-loop perturbative argument. When all the eigenvalues are well separated from each other, one can expand the matrices A_μ and ψ_α around diagonal matrices as

$$(A_\mu)_{ij} = x_{i\mu}\,\delta_{ij} + a_{\mu ij}\,; \qquad (\psi_\alpha)_{ij} = \xi_{i\alpha}\,\delta_{ij} + \varphi_{\alpha ij}\,, \tag{4}$$

where a_μ and φ_α are matrices containing only off-diagonal elements. One can then integrate over the off-diagonal elements a_μ and φ_α up to one-loop, by fixing the gauge properly and including the Faddeev-Popov ghosts. The integration over the bosonic off-diagonal elements a_μ and the Faddeev-Popov ghosts, gives a logarithmic attractive potential between all the pairs of x_i [14]. This potential, however, is exactly cancelled by the contribution of the fermionic off-diagonal elements φ_α due to supersymmetry. This cancellation is responsible for the cluster property of the

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model [2], which is important for the interpretation of the model as a string theory. In order to calculate the effective potential for $x_{i\mu}$, one still has to integrate over the fermionic diagonal elements $\xi_{i\alpha}$, which is nontrivial. In ref. [8], it has been shown that the effective potential for $x_{i\mu}$ can be given by a branched-polymer like attractive interaction among N points in D-dimensional space-time represented by $x_{i\mu}$ ($i = 1, \ldots, N$). Now counting the power of x_i in the partition function, the integration measure for $x_{i\mu}$ gives $D\,(N-1)$ and the integration over $\xi_{i\alpha}$ gives $-\frac{3}{2}\,p\,(N-1)$, where $p = 2\,(D-2)$ is the number of real components of the spinor considered for each D. Therefore, naively one concludes that when

$$D\,(N-1) - \frac{3}{2}\,p\,(N-1) = 2\,(3-D)\,(N-1) \geq 0\,, \tag{5}$$

the integration over $x_{i\mu}$ is divergent and otherwise convergent. This means that the integral is divergent for $D = 3$ and convergent for $D = 4, 6, 10$, irrespectively of N. This conclusion is in agreement with an exact result available for $N = 2$ [7] and a numerical result obtained for $N = 3$ [9]. For $D = 4$, Monte Carlo simulations show that the conclusion extends to the large-N limit [10]. Therefore, it is conceivable that the above conclusion obtained by the one-loop argument holds in general, although it is not a rigorous proof.

Going back to the definition of the model (1), let us first integrate over the fermionic matrices ψ_α, which induces an effective action for A_μ. When fermions are real Grassmann variables as in the present $D = 10$ case, the fermion integral yields a pfaffian of a $p\,(N^2-1) \times p\,(N^2-1)$ matrix. On the other hand, when fermions are complex Grassmann variables as in the $D = 4, 6$ case, the fermion integral yields a determinant of a $p\,(N^2-1)/2 \times p\,(N^2-1)/2$ matrix. For $D = 4$, the fermion determinant is real positive, which allows a direct Monte Carlo study of the model [10]. For $D = 6, 10$, the fermion determinant ($D = 6$) or pfaffian ($D = 10$) is complex in general, since fermions are essentially chiral in these cases. This causes the notorious sign problem. There are some exceptions[4] when N is small [7, 9]. For $D = 6$, the fermion determinant is real positive for $N = 2$. For $D = 10$, the pfaffian is real positive for $N = 2$ and real (but not positive definite) for $N = 3$. Although we consider only the $D = 10$ case in what follows, the analysis can be readily applied to the $D = 6$ case as well. We also restrict ourselves to $N \geq 4$ so that the pfaffian is complex generically.

In $D = 10$, the fermion integral yields

$$Z_f[A] = \int \mathrm{d}\psi \mathrm{e}^{-S_f} = \mathrm{Pf}\ \mathcal{M}\,, \tag{6}$$

[4] This is because, when (N^2-1) is smaller than D, one can always make a D-dimensional rotation to set $A_\mu = 0$ for $\mu = 1, \ldots, D-(N^2-1)$, since one can regard A_μ^a as (N^2-1), D-dimensional real vectors. The statements in $D = 10$ for $N = 2$ and $N = 3$, for example, follow from (14) and (13), respectively.

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where

$$\mathcal{M}_{a\alpha,b\beta} = -i\, f_{abc}(\, \mathcal{C}\, \Gamma_\mu)_{\alpha\beta} A^c_\mu \tag{7}$$

is a $16\,(N^2-1) \times 16\,(N^2-1)$ anti-symmetric matrix, regarding each of $(a\alpha)$ and $(b\beta)$ as a single index. The real totally-antisymmetric tensor f_{abc} gives the structure constants of SU(N). In what follows, it proves convenient to work with an explicit representation of the gamma matrices given by

$$\begin{aligned}
&\Gamma_1 = i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2\,, && \Gamma_2 = i\sigma_2 \otimes \sigma_2 \otimes \mathbf{1} \otimes \sigma_1\,, && \Gamma_3 = i\sigma_2 \otimes \sigma_2 \otimes \mathbf{1} \otimes \sigma_3\,,\\
&\Gamma_4 = i\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbf{1}\,, && \Gamma_5 = i\sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbf{1}\,, && \Gamma_6 = i\sigma_2 \otimes \mathbf{1} \otimes \sigma_1 \otimes \sigma_2\,,\\
&\Gamma_7 = i\sigma_2 \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_2\,, && \Gamma_8 = i\sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\,, && \Gamma_9 = i\sigma_3 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\,,\\
&\Gamma_{10} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\,,
\end{aligned} \tag{8}$$

for which the charge conjugation matrix $\mathcal{C}$ becomes a unit matrix.

Some important properties of the pfaffian Pf $\mathcal{M}(A)$ are in order. First of all, it is invariant under a ten-dimensional Lorentz transformation, $A'_\mu = \Lambda_{\mu\nu} A_\nu$, where Λ is an SO(10) matrix. It is also invariant under a gauge transformation $A'_\mu = g A_\nu g^\dagger$, where g is an SU(N) matrix. Under a parity transformation,

$$A^P_{10} = -A_{10}\,; \qquad A^P_i = A_i \qquad (\text{for } i = 1, \ldots, 9)\,, \tag{9}$$

the pfaffian becomes complex conjugate,

$$\text{Pf } \mathcal{M}(A^P) = \{\text{Pf } \mathcal{M}(A)\}^*\,, \tag{10}$$

since $\mathcal{M}(A^P) = \mathcal{M}(A)^*$. As a consequence, the pfaffian Pf $\mathcal{M}(A)$ is real, when $A_{10} = 0$. This also means that $\det \mathcal{M} = (\text{Pf } \mathcal{M})^2$ is real positive, when $A_{10} = 0$.

We next consider the case with $A_9 = A_{10} = 0$. In this case, the pfaffian is actually equal to the fermion determinant of eight-dimensional Weyl fermion which we denote as $\det \mathcal{M}^{(8)}$. $\mathcal{M}^{(8)}$ is an $8\,(N^2-1) \times 8\,(N^2-1)$ matrix defined as $\mathcal{M}^{(8)}_{a\alpha,b\beta} = -i\, f_{abc}(\tilde{\Gamma}_\mu)_{\alpha\beta} A^c_\mu$, where the eight-dimensional gamma-matrices $\tilde{\Gamma}_\mu$ after Weyl projection are given, for example, by

$$\begin{aligned}
&\tilde{\Gamma}_1 = i\sigma_2 \otimes \sigma_2 \otimes \sigma_2\,, && \tilde{\Gamma}_2 = i\sigma_2 \otimes \mathbf{1} \otimes \sigma_1\,, && \tilde{\Gamma}_3 = i\sigma_2 \otimes \mathbf{1} \otimes \sigma_3\,,\\
&\tilde{\Gamma}_4 = i\sigma_1 \otimes \sigma_2 \otimes \mathbf{1}\,, && \tilde{\Gamma}_5 = i\sigma_3 \otimes \sigma_2 \otimes \mathbf{1}\,, && \tilde{\Gamma}_6 = i\mathbf{1} \otimes \sigma_1 \otimes \sigma_2\,,\\
&\tilde{\Gamma}_7 = i\mathbf{1} \otimes \sigma_3 \otimes \sigma_2\,, && \tilde{\Gamma}_8 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\,.
\end{aligned} \tag{11}$$

With this representation, one finds that $\mathcal{M}^{(8)}$ is a real matrix, which means that $\det \mathcal{M}^{(8)}$ is real. However, it is not positive definite, as we have checked numerically. In fact, this is the case even for $A_8 = 0$. However, if we take $A_7 = A_8 = 0$, it becomes positive definite, since in this case, we have $\det \mathcal{M}^{(8)} = |\det \mathcal{M}^{(6)}|^2$, where $\det \mathcal{M}^{(6)}$ is the fermion determinant of six-dimensional Weyl fermion [9].

Finally, when $A_3 = A_4 = \cdots = A_{10} = 0$, one finds that Pf $\mathcal{M}(A) = 0$ [9]. This can be proved in the following way. First we note that $(\text{Pf } \mathcal{M})^2 = \det \mathcal{M} = |\det U|^{16}$,

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where U is an $(N^2-1)\times(N^2-1)$ matrix defined as $U_{ab} = f_{abc}X^c$, where $X^c = A_1^c + iA_2^c$. Since $U_{ab}X^b = 0$, the matrix U has a zero-eigenvalue, and therefore $\det U = 0$. This completes the proof.

Using Lorentz invariance of the pfaffian $\mathrm{Pf}\,\mathcal{M}(A)$, we summarize some of its important properties as follows. Let us first define sets of "degenerate" configurations Ω_d as

$$\Omega_d = \left\{ \{A_\mu\}\,;\; n_\mu^{(i)} A_\mu = 0 \;\text{ for }\; \exists\, n_\mu^{(i)} (i = 1, \ldots, 10-d) \;\text{ linearly independent} \right\}. \quad (12)$$

Obviously, $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_9$. Then the statements are

$$(a) \;\text{ When } \{A_\mu\} \in \Omega_9\,, \qquad \mathrm{Pf}\;\mathcal{M}(A) \in \mathbb{R}\,. \quad (13)$$

$$(b) \;\text{ When } \{A_\mu\} \in \Omega_6\,, \qquad \mathrm{Pf}\;\mathcal{M}(A) \geq 0\,. \quad (14)$$

$$(c) \;\text{ When } \{A_\mu\} \in \Omega_2\,, \qquad \mathrm{Pf}\;\mathcal{M}(A) = 0\,. \quad (15)$$

Here we recall that in the IIB matrix model, the space-time is treated as a dynamical object represented by the bosonic matrices A_μ. In this regard, generic configurations in Ω_d describe d-dimensional space-time. It is very suggestive that the phase of the pfaffian is sensitive to the dimensionality d of the space-time. Indeed, the above results will play a crucial role in the following analysis.

Let us write the effective action induced by fermion integral as $\Gamma_{\rm eff} = -\ln(\mathrm{Pf}\,\mathcal{M})$, which we decompose into the real part $\Gamma^{(r)}$ and the imaginary part $\Gamma^{(i)}$ as $\Gamma_{\rm eff} = \Gamma^{(r)} + i\,\Gamma^{(i)}$. Then we generalize the theory by introducing an integer parameter ν, which couples only to $\Gamma^{(i)}$, the imaginary part of the effective action, as

$$Z_\nu = \int \mathrm{d}A \mathrm{e}^{-S_b[A]-\Gamma^{(r)}[A]} \mathrm{e}^{-i\,\nu\,\Gamma^{(i)}[A]}\,. \quad (16)$$

The parameter ν has to be an integer since $\Gamma^{(i)}$ is defined only up to modulo 2π. For $\nu = 1$, the model reduces to the original model. Since $\Gamma^{(i)}$ flips its sign under a parity transformation due to (10), the models with ν and $-\nu$ are nothing but parity partners. Since $\Gamma^{(r)}$ and $\Gamma^{(i)}$ are both Lorentz invariant and gauge invariant separately, the generalized model is also both Lorentz invariant and gauge invariant. According to the one-loop perturbative argument given below (4), the deformed IIB matrix model continues to be well-defined for arbitrary ν, and moreover, it preserves the cluster property, which is an important consequence of the supersymmetry of the original model.

Another important point about the generalization we consider is that $\langle S_b \rangle$ can be calculated exactly and the result is independent of the parameter ν. For this, we recall that in the original supersymmetric large-N reduced models ($D = 4, 6, 10$), one obtains

$$\langle S_b \rangle = \frac{1}{4}\left(D + \frac{p}{2}\right)(N^2-1) = \frac{1}{2}(D-1)(N^2-1)\,, \quad (17)$$

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using a scaling argument similar to [11]. Due to the fact that $\Gamma^{(i)}[A]$ is invariant under a scale transformation of A_μ, one can easily find that the above result is unaltered by the generalization to arbitrary ν for $D = 6, 10$. In particular, this holds true even in the $\nu \to \infty$ limit. Note, however, that this does not necessarily imply that all the vacuum expectation values of the generalized model are independent of ν. Even their convergence in the $\nu \to \infty$ limit is nontrivial. This point shall be clarified later.

We prove a property of $\Gamma^{(i)}$, which turns out to be essential in the analysis of the $\nu \to \infty$ limit. We first recall that the pfaffian $\mathrm{Pf}\,\mathcal{M}(A)$ is a polynomial of A^a_μ of order $8\,(N^2-1)$. Hence, $\Gamma^{(i)}$ is infinitely differentiable at a configuration, for which the matrix $\mathcal{M}$ is invertible. When the configuration belongs to Ω_d ($d = 1, \ldots, 8$), we find that

$$\frac{\partial^n \Gamma^{(i)}}{\partial A^{a_1}_{\mu_1} \partial A^{a_2}_{\mu_2} \cdots \partial A^{a_n}_{\mu_n}} = -\frac{1}{2} \frac{\partial^n}{\partial A^{a_1}_{\mu_1} \partial A^{a_2}_{\mu_2} \cdots \partial A^{a_n}_{\mu_n}} \mathrm{Im} \ln \det \mathcal{M} = 0\,, \qquad \text{for } n = 1, \ldots, (9-d)\,. \tag{18}$$

This is because, up to $(9-d)$-th order of perturbations, the configuration stays within Ω_9, and therefore $\det \mathcal{M}$ remains to be real positive due to (13).

In the $\nu \to \infty$ limit, the integration over A_μ is dominated by the configurations which satisfy the saddle-point equation given by

$$0 = \frac{\partial \Gamma^{(i)}}{\partial A^a_\mu} = -\frac{1}{2} \frac{\partial}{\partial A^a_\mu} \mathrm{Im} \ln \det \mathcal{M} = -\frac{1}{2} \mathrm{Im}\, \mathrm{Tr} \left(\mathcal{M}^{-1} \mathcal{B}^{(\mu a)} \right), \tag{19}$$

where $\mathcal{B}^{(\mu a)}$ is an A-independent $16\,(N^2-1) \times 16\,(N^2-1)$ matrix defined by

$$\left(\mathcal{B}^{(\mu a)} \right)_{b\beta, c\gamma} = \frac{\partial\, \mathcal{M}_{b\beta, c\gamma}}{\partial A^a_\mu} = -i\, f_{abc} (\mathcal{C}\, \Gamma_\mu)_{\beta\gamma}\,. \tag{20}$$

Since the equation (19) gives $10\,(N^2-1)$ constraints among $10\,(N^2-1)$ real variables, naively one would expect that only a very small portion of the configuration space survives. This is not the case, however. In fact, we find that all the configurations that belong to Ω_8 satisfy the saddle-point equation (19) due to (18). The existence of solutions to (19) other than of this type, cannot be excluded. It is reasonable, however, to consider that the configurations in Ω_8 dominate in the sense of Lebesgue measure. Hence, at $\nu = \infty$ the full ten-dimensional Lorentz invariance is broken down at least to an eight-dimensional one. In order to examine whether the remaining eight-dimensional Lorentz invariance further breaks down, we still have to integrate over the saddle-point configurations. An important point here is that the imaginary part of the effective action $\Gamma^{(i)}$ for the saddle-point configurations takes only 0 or π due to (13). This means that there are actually two sequences of ν, (I) $\nu = 0, 2, 4, \ldots, 2\infty$ and (II) $\nu = 1, 3, 5, \ldots, (2\infty + 1)$, which give two *a priori* different limiting theories.

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In order to formulate the integration over the saddle-point configurations, we first rotate them so that they satisfy $A_9 = A_{10} = 0$ and then integrate over $A_1, A_2, \ldots, A_8$. We define the Hesse matrix for $\Gamma^{(i)}[A]$ within the directions transverse to the integration domain as

$$H_{ja,kb} = \frac{\partial^2 \Gamma^{(i)}}{\partial A_j^a \partial A_k^b} = -\frac{1}{2} \frac{\partial^2}{\partial A_j^a \partial A_k^b} \operatorname{Im} \ln \det \mathcal{M} = \frac{1}{2} \operatorname{Im} \operatorname{Tr} \left(\mathcal{M}^{-1} \mathcal{B}^{(ja)} \mathcal{M}^{-1} \mathcal{B}^{(kb)} \right), \tag{21}$$

where $j, k = 9, 10$. We find that the Hesse matrix $H_{ja,kb}$ has 16 zero-eigenvalues with eigenvectors corresponding to perturbations

$$(\delta A_9^a, \delta A_{10}^a) = (A_1^a, 0), (A_2^a, 0), \ldots, (A_8^a, 0), (0, A_1^a), (0, A_2^a), \ldots, (0, A_8^a). \tag{22}$$

These zero-modes are a reflection of the fact that the configurations after these perturbations still stay within Ω_8, and thus satisfy the saddle-point equation (19). When N is even, the Hesse matrix $H_{ja,kb}$ has actually two more zero-eigenvalues. To see this, we first note that $H_{9a,9b} = H_{10a,10b} = 0$ and $H_{9a,10b} = H_{10b,9a}$. Note also that $H_{9a,10b} = -H_{9b,10a}$ due to (10), which means that an $(N^2-1) \times (N^2-1)$ matrix K_{ab} defined by $K_{ab} = H_{9a,10b}$ is antisymmetric. Due to a general property of an antisymmetric matrix of odd size, K_{ab} for even N should have a zero-eigenvalue, whose corresponding eigenvector we denote as χ^a. Then one finds that $(\delta A_9^a, \delta A_{10}^a) = (\chi^a, 0), (0, \chi^a)$ are eigenvectors of the Hesse matrix $H_{ja,kb}$ with zero-eigenvalues. Unlike the 16 zero-modes in (22), these two additional zero-modes, which exist only for even N, have nothing to do with the symmetry of the space of solutions to the saddle-point equation (19). Hence, they should be considered as accidental zero-modes. Since the saddle-point analysis in such a case becomes more complicated, we restrict ourselves to the odd N case in what follows.

In order to deal with the 16 zero-modes, which actually correspond to the Lorentz transformation in the $(i, 9)$ and $(i, 10)$ planes ($i = 1, 2, \ldots, 8$), we have to take into account the phase volume analogous to the Faddeev-Popov determinant. In the present case, it can be given by the 16-dimensional volume spanned by the 16 vectors (22) in the configuration space. We denote this phase volume as $\mathcal{V}^{(16)}$. Thus we arrive at the models, which describe the two limiting theories corresponding to the even/odd ν sequences,

$$Z_{\nu=2\infty} = \int \left(\prod_{\mu=1}^{8} \mathrm{d}A_\mu \right) \mathcal{V}^{(16)} \, |J|^{-1/2} \, \mathrm{e}^{-S_b} \left| \det \mathcal{M}^{(8)} \right|, \tag{23}$$

$$Z_{\nu=2\infty+1} = \int \left(\prod_{\mu=1}^{8} \mathrm{d}A_\mu \right) \mathcal{V}^{(16)} \, |J|^{-1/2} \, \mathrm{e}^{-S_b} \, \det \mathcal{M}^{(8)}, \tag{24}$$

where J denotes the determinant of the Hesse matrix $H_{ja,kb}$ after removing the zero-

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modes. We recall that $\det \mathcal{M}^{(8)}$ is real but not positive definite, which makes the two models *a priori* different. One can see that the exact result (17) for $\langle S_b \rangle$ can be reproduced from (23) and (24) as it should, by noting that $\mathcal{V}^{(16)} \mapsto \lambda^{16} \mathcal{V}^{(16)}$ and $J \mapsto \lambda^{-2\{2(N^2-1)-16\}} J$ under a scale transformation $A_\mu \mapsto \lambda A_\mu$.

Let us consider the question whether the remaining eight-dimensional Lorentz invariance of the models (23) and (24) is further broken. For this we show that the hessian J becomes zero for configurations in Ω_7 and that the order of zero increases for configurations in Ω_d with smaller d. To quantify this statement, let us consider a configuration with $A_1, \ldots, A_d$ being generic and $A_{d+1}, \ldots, A_8$ being of order ϵ. We first note that each element of $H_{ja,kb}$ for such a configuration becomes of order $\epsilon^{(8-d)}$ due to (18). Therefore, the non-zero eigenvalues of $H_{ja,kb}$ become of order $\epsilon^{(8-d)}$. We have checked numerically that they are not of order higher than $\epsilon^{(8-d)}$ generically. This means that J is of order $\epsilon^{(8-d)\,\{2\,(N^2-1)-16\}}$. On the other hand, degenerate configurations with smaller d is suppressed by the entropy factor $\epsilon^{(8-d)\,(N^2-1)}$ coming from the integration measure. The phase volume factor $\mathcal{V}^{(16)}$ gives also a suppression of order $\epsilon^{2\,(8-d)}$. Collecting all the powers of ϵ, we find a suppression of order $\epsilon^{10\,(8-d)}$. This means, first of all, that the integrals (23) and (24) are non-singular at the degenerate configurations. Secondly, we note that, as far as the N-dependent part is concerned, the suppression for smaller d coming the entropy factor $\epsilon^{(8-d)\,(N^2-1)}$ is exactly cancelled by the enhancement coming from $|J|^{-1/2}$. This means that there is essentially no entropical barrier against having smaller d.

So far, we have been focusing on the effect of $\Gamma^{(i)}$ on the dynamics of the IIB matrix model. The effect of S_b and $\Gamma^{(r)}$, on the other hand, has been studied in ref. [8] by using the low-energy effective theory obtained by the one-loop perturbation theory (4). There, it was argued that the complicated branched-polymer interaction among the diagonal elements $x_{i\mu}$ of the bosonic matrices A_μ might induce a collapse of the distribution of $x_{i\mu}$. Monte Carlo simulation of the low-energy effective theory at $\nu = 0$ [13] shows that the effect of S_b and $\Gamma^{(r)}$ is not sufficient to induce a spontaneous breakdown of Lorentz invariance. However, after taking into account the effect of $\Gamma^{(i)}$ by sending ν to infinity, it is very plausible that the remaining eight-dimensional Lorentz invariance of the models (23) and (24) is further broken by the effect of S_b and $\Gamma^{(r)}$, because there is no entropical barrier against having degenerate configurations any more. If this is the case, the vacuum at $\nu = \infty$ is given by degenerate configurations with $d < 8$. In this regard, we recall also that the pfaffian $\mathrm{Pf}\,\mathcal{M}(A)$ is zero when $\{A_\mu\} \in \Omega_2$, as stated in (15). Therefore, the dimensionality d of the vacuum configurations must be $d > 2$.

In order to check the above statements and to determine the dimensionality d of the vacuum configurations at $\nu = \infty$, one has to carry out the integration over the saddle-point configurations described by (23) and (24), for example, by Monte Carlo simulation. Note that the model (23) is not plagued by the sign problem any more. The model (24), on the other hand, can be studied using the configurations

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generated with the model (23) as

$$\langle \mathcal{O} \rangle_{\nu=2\infty+1} = \frac{\left\langle \mathcal{O}\, \mathrm{sgn}(\det \mathcal{M}^{(8)}) \right\rangle_{\nu=2\infty}}{\langle \mathrm{sgn}(\det \mathcal{M}^{(8)}) \rangle_{\nu=2\infty}} \,. \tag{25}$$

If the fermion determinant $\det \mathcal{M}^{(8)}$ does not have a definite sign for dominant configurations, the denominator as well as the numerator of the r.h.s. in (25) becomes very small, exhibiting the sign problem. However, if it turns out that the vacuum of the model (23) is given by configurations with $d \leq 6$, then due to (14), we have $\mathrm{sgn}(\det \mathcal{M}^{(8)}) = 1$ for dominant configurations. In this case, the sign problem, which *a priori* exists in the model (24), is solved in a sense *dynamically*, and the model (24) is actually equivalent to the model (23), since $\langle \mathcal{O} \rangle_{\nu=2\infty+1} = \langle \mathcal{O} \rangle_{\nu=2\infty}$ for any observables $\mathcal{O}$.

To summarize, we considered a deformation of the IIB matrix model by introducing an integer parameter ν which couples to $\Gamma^{(i)}$, the imaginary part of the effective action induced by fermions. We studied the deformed model at $\nu = \infty$, where the integration over the bosonic matrices is dominated by the configurations for which $\Gamma^{(i)}$ is stationary. First of all, there is still a huge configuration space left as the saddle-point configurations which we have to integrate over. Secondly, these saddle-point configurations have more than two shrunken directions. Thirdly, the more shrunken directions the configuration has, the more stationary $\Gamma^{(i)}$ becomes. This gives rise to an enhancement for configurations with more shrunken directions, and the effect was shown to cancel exactly the N-dependent entropical barrier against having such configurations. An intriguing feature of this enhancement is that it occurs exactly when the configuration becomes a lower-dimensional hyperplane. This may be responsible for generating a *flat* space-time instead of a curved one or a fractal one, as a result of the dynamics of the IIB matrix model. We argued that the dimensionality d of the space-time generated dynamically in the deformed IIB matrix model at $\nu = \infty$ is $2 < d \leq 8$ and most likely $d < 8$. We derived the models which describe the integration over the saddle-point configurations for the two *a priori* different limiting theories corresponding to the even/odd ν sequences. Remarkably, the model with the even ν sequence is not plagued by the sign problem. One can therefore study the model by standard Monte Carlo simulation to extract the dimensionality d of the vacuum configurations. If this turns out to be $d \leq 6$, the model with the odd ν sequence is actually equivalent to the model with the even ν sequence.

Whether the original model ($\nu = 1$) belongs to the same phase as $\nu = \infty$ is a nontrivial question, which is not accessible through standard Monte Carlo simulation due to the sign problem. We quote, however, an example from history, where an analogous approach was successful. It is the strong coupling limit in the lattice formulation of nonabelian gauge theories. Although one should send the bare coupling constant to zero in the continuum limit, confinement in these theories has been clearly demonstrated in the strong coupling limit [15]. The limit also provides a qualitative

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understanding of the spontaneous breakdown of chiral symmetry [16]. The existence of such an approach is indeed one of the advantages of having a nonperturbative formulation, which should also apply to the case of string theory. Note, however, that the above successes in the case of nonabelian gauge theories rely crucially on the fact that there is no phase transition between the strong coupling regime and the weak coupling regime [17]. An illustrative example clarifying this point is that confinement holds in the strong coupling limit even for abelian gauge theories, for which there is actually a phase transition to a deconfining phase at an intermediate coupling constant. The fact that the model at $\nu = \infty$ still has a huge configuration space to integrate over, which is quite peculiar to this system, may suggest that it is in fact quite close to the original model ($\nu = 1$). Although it is hard to justify this statement rigorously, it would be certainly worthwhile to explore further the dynamics of the deformed IIB matrix model in this limit.

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References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [`hep-th/9610043`].

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [`hep-th/9612115`].

[3] M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, *String field theory from IIB matrix model*, *Nucl. Phys.* **B 510** (1998) 158 [`hep-th/9705128`].

[4] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063.

[5] L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [`hep-th/9906064`].

[6] J. Greensite and F.R. Klinkhamer, *Superstring amplitudes and contact interactions*, *Nucl. Phys.* **B 304** (1988) 108;
V.A. Kostelecký and S. Samuel, *Spontaneous breaking of Lorentz symmetry in*

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string theory, *Phys. Rev.* **D 39** (1989) 683; *Gravitational phenomenology in higher-dimensional theories and strings*, *Phys. Rev.* **D 40** (1989) 1886;
V.A. Kostelecký and R. Potting, *CPT and strings*, *Nucl. Phys.* **B 359** (1991) 545; *Expectation values, Lorentz invariance, and CPT in the open bosonic string*, *Phys. Lett.* **B 381** (1996) 89 [hep-th/9605088];
V.A. Kostelecký, M.J. Perry and R. Potting, *Off-shell structure of the string sigma model*, hep-th/9912243.

[7] T. Suyama and A. Tsuchiya, *Exact results in $N_c = 2$ IIB matrix model*, *Prog. Theor. Phys.* **99** (1998) 321 [hep-th/9711073].

[8] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, *Space-time structures from IIB matrix model*, *Prog. Theor. Phys.* **99** (1998) 713 [hep-th/9802085].

[9] W. Krauth, H. Nicolai and M. Staudacher, *Monte Carlo approach to M-theory*, *Phys. Lett.* **B 431** (1998) 31 [hep-th/9803117].

[10] J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *large-N dynamics of dimensionally reduced 4d* SU(N) *super Yang-Mills theory*, hep-th/0003208.

[11] T. Hotta, J. Nishimura and A. Tsuchiya, *Dynamical aspects of large-N reduced models*, *Nucl. Phys.* **B 545** (1999) 543 [hep-th/9811220].

[12] W. Bietenholz, A. Pochinsky and U.J. Wiese, *Meron cluster simulation of the theta vacuum in the 2D $O(3)$ model*, *Phys. Rev. Lett.* **75** (1995) 4524 [hep-lat/9505019];
S. Chandrasekharan and U.-J. Wiese, *Meron-cluster solution of a fermion sign problem*, *Phys. Rev. Lett.* **83** (1999) 3116 [cond-mat/9902128].

[13] J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, in preparation.

[14] G. Bhanot, U.M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett.* **B 113** (1982) 47.

[15] K.G. Wilson, *Confinement of quarks*, *Phys. Rev.* **D 10** (1974) 2445.

[16] H. Kluberg-Stern, A. Morel, O. Napoly and B. Petersson, *Spontaneous chiral symmetry breaking for a* U(N) *gauge theory on a lattice*, *Nucl. Phys.* **B 190** (1981) 504;
N. Kawamoto and J. Smit, *Effective lagrangian and dynamical symmetry breaking in strongly coupled lattice QCD*, *Nucl. Phys.* **B 192** (1981) 100.

[17] M. Creutz, *Monte Carlo study of quantized* SU(2) *gauge theory*, *Phys. Rev.* **D 21** (1980) 2308.

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